

# A novel renormalizable representation of the Yang-Mills theory

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## Abstract

For a generic gauge-invariant correlator  $\langle Q[A_\mu] \rangle_A$ , we reformulate the standard  $D = 4$  Yang-Mills theory as a renormalizable system of two interacting fields  $a_\mu$  and  $B_\mu$  which faithfully represent high- and low-energy degrees of freedom of the single gauge field  $A_\mu$  in the original formulation. It opens a possibility to synthesize an *infrared-nonsingular* weak-coupling series, employed to integrate over  $a_\mu$  for a given background  $B_\mu$ , with qualitatively different methods. These methods are to be applied to evaluate the resulting (after the  $a_\mu$ -integration) representation of  $\langle Q[A_\mu] \rangle_A$  in terms of gauge-invariant generically non-local low-energy observables, like Wilson loops. The latter observables are averaged over  $B_\mu$  with respect to a *gauge-invariant* Wilsonian effective action  $S_{eff}[B]$ . To avoid a destructive dissipation between the high- and low-energy excitations, we implement a specific fine-tuning of the interaction between the pair of the fields: prior to the integration over  $B_\mu$ , the expectation value  $\langle a_\mu \rangle_a$  vanishes, in the tree order of the loop-wise expansion, for an arbitrary configuration of  $B_\mu$ .

Keywords: Yang-Mills, Wilsonian action, gauge invariance

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# 1 Introduction

The  $D = 4$  dimensional  $SU(N)$  Yang-Mills theory ( $YM_D$ ), defined by the action

$$S_{YM}[A_\mu] = \int d^D x \operatorname{tr} \left( F_{\mu\nu}^2(A) \right) / 4g^2, \quad (1.1)$$

belongs to a class of the systems where the physics at short distances, characterized by asymptotic freedom, is qualitatively different compared to the low-energy physics governed by confinement. Therefore, it is reasonable to search for a formalism which allows to effectively combine different computational techniques applied respectively to the high- and low-energy dynamics interpolated at a scale  $\Lambda_{int}$  sufficiently larger than<sup>2</sup>  $\Lambda_{YM}$ .

The good old weak-coupling series are known to be well-defined only in the ultraviolet ( $UV$ ) domain of relative distances sufficiently smaller than  $\Lambda_{YM}^{-1}$ . In the  $UV$  domain, the series can be extended including an input of the infrared ( $IR$ ) dynamics of the system. For this purpose, the only theoretical method so far is the operator product expansion ( $OPE$ ). Various implementations of  $OPE$  synthesize the weak-coupling series with matrix elements of local operators which parameterize the  $IR$  input in question. Unfortunately, as well as the series itself, the language of local operators is not robust enough to successfully apply this method to processes dominated by the large-distance phenomena like confinement implying a string-like pattern of the excitations. It calls to push the idea of the synthesis even further so that the weak-coupling expansion, in effect being restricted to the description of the short-distance physics, is properly combined with a description of the  $IR$  phenomena by gauge-invariant *non-local* correlators like Wilson loops.

The aim of the present paper is to propose, in the Euclidean space, such a formalism where the latter correlators are averaged directly with respect to a gauge-invariant *Wilsonean* effective action which describes a strongly coupled gauge system representing the low-energy dynamics of the theory (1.1). For this purpose, we reformulate the theory (1.1) as a system of two interacting fields  $a_\mu$  and  $B_\mu$  which, being described by a certain auxiliary action  $\tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu]$  *renormalizable* from the power counting viewpoint, represent the high- and low-energy modes of  $A_\mu$  respectively.

To accomplish a generic reformulation of any given gauge-invariant correlator  $\langle \mathcal{Q}[A_\mu] \rangle_A$  in the theory (1.1) in terms of a pair of fields, we introduce a judicious Faddeev-Popov unity as a functional which depends on a dynamical field  $B_\mu$ . The corresponding gauge condition is imposed on the combination  $A_\mu - B_\mu$  to be identified with the field  $a_\mu$ . As a result,  $\langle \mathcal{Q}[A_\mu] \rangle_A$  is rewritten in the form

$$\langle \mathcal{Q}[A_\mu] \rangle_A = \left\langle \left\langle \mathcal{Q}[a_\mu + B_\mu] \right\rangle_a^{B^{ext}} \right\rangle_B, \quad (1.2)$$

where the averaging  $\langle .. \rangle_a^B$  over the high-energy field  $a_\mu$  is performed, for a given low-energy field  $B_\mu \equiv B_\mu^{ext}$  considered as external, with the "microscopic" action  $\tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu]$ . It allows to compute both the various averages  $\langle .. \rangle_a^B$  and the associated partition function  $Z_1[B]$  (of the auxiliary high-energy theory for a fixed  $B_\mu$ ) using the  $1/N$  weak-coupling series running in the renormalized coupling constant  $g_r^2 \equiv g^2(\Lambda/\Lambda_{YM})$  associated with a scale  $\Lambda$ . In particular, to avoid a destructive dissipation between the high- and low-energy excitations, the interaction between the pair of the fields has to be judiciously constrained. We impose that, at least in the tree-order of the renormalized loop-wise expansion in the external background  $B_\mu \equiv B_\mu^{ext}$ , the constraint

$$\langle a_\mu(\mathbf{x}) \rangle_a^{B^{ext}} = 0, \quad \forall B_\mu, \quad (1.3)$$

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<sup>2</sup>In the theory (1.1), the perturbative solution  $g_r^2(\Lambda/\Lambda_{YM})$  of the renormgroup equation blows up at the scale  $\Lambda = \Lambda_{YM}$ .

holds true for a *generic* configuration of  $B_\mu$ . In the high-energy sector, it is shown to maintain that the spurious 'symmetry breaking' (displayed by  $\langle a_\mu(\mathbf{x}) \rangle_a^B \neq 0$ ), being suppressed by powers of the coupling constant  $g_r^2$ , leaves the background perturbation theory well-defined.

The subsequent integration  $\langle .. \rangle_B$  over the low-energy field  $B_\mu$  is performed with respect to the corresponding effective action  $S_{eff}[B_\mu]$  conventionally defined by the relation

$$\exp(-S_{eff}[B_\mu]) = Z_1[B] , \quad (1.4)$$

where the partition function  $Z_1[B]$  is introduced above. To maintain a gauge-invariant description of the low-energy phenomena, the action  $\tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu]$  is imposed to be invariant under the background gauge transformations [4, 5]:

$$B_\mu \longrightarrow B_\mu^{(\psi)} = U(\psi)(B_\mu + i\partial_\mu)U^{-1}(\psi) \quad , \quad a_\mu \longrightarrow a_\mu^{(\psi)} = U(\psi)a_\mu U^{-1}(\psi) , \quad (1.5)$$

which entails that the associated effective action  $S_{eff}[B]$  respects the gauge symmetry. In turn, it implies that, after the integration over  $a_\mu$ , the average (1.2) is indeed expressed in terms of (non-local) gauge-invariant correlators depending on  $B_\mu$ . Although it is definitely beyond the scope of the paper to discuss a scheme for evaluation of the low-energy correlators  $\langle .. \rangle_B$ , we note that these correlators may be approached employing the stringy form of the  $1/N$  strong-coupling expansion [2] (see also [3]) yielding a continuum counterpart of the corresponding lattice expansion. In this way, the reformulation (1.2) of (1.1) is suggested to provide a bridge to interpolate between the  $1/N$  weak- and  $1/N$  strong-coupling series.

Next, the resolution of the constraint (1.3), truncated to a given order of the loop-wise expansion, is to be understood in the context of the following prescription that takes advantage of the freedom in the choice of  $\tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu]$ . To begin with, it is convenient to impose that the transformation (1.2) results in the axial gauge condition for  $a_\mu$ :

$$f^c(a, B) = n_\mu a_\mu^c = 0 \quad , \quad n_\mu^2 = 1 , \quad (1.6)$$

where  $n_\mu$  is a constant  $D$ -vector, and we presume that  $n_\mu a_\mu = a_0 = 0$  which leaves  $D - 1$  dynamical components  $a_i$ . In addition to Eq. (1.6), we impose that the difference  $\tilde{S}_{\Lambda_\varepsilon}[a_i, B_\mu] - S_{YM}[a_\mu + B_\mu]|_{a_0=0}$  defines a Lagrangian which is a quadratic polynomial in  $a_i$  with generically  $B_\mu$ -dependent coefficients. The quadratic in  $a_i$  term serves merely to attribute, at the tree-level, a mass ( $\sim \Lambda_{int}$ ) to the  $a_i$ -field which, in turn, facilitates the implementation of the transformation (1.2) as a multiscale decomposition. As a by-product, for sufficiently large value of the interpolation scale  $\Lambda_{int}$ , the background perturbation theory is free of spurious  $IR$  divergences. It is also noteworthy that, akin to the case of the original theory (1.1), there are only *two* propagating polarizations of the field  $a_i$  (as it is formalized by eq. (5.1)). Concerning the linear in  $a_i$  term, it is completely determined by the constraint (1.3) or its  $L$ -loop truncation. E.g., in the leading tree-order of the loop-wise expansion, Eq. (1.3) reduces to the requirement that, for *any*  $B_\mu$ , the tree-level approximation<sup>3</sup>  $\tilde{S}^{tr}[a_i, B_\mu]$  to  $\tilde{S}_{\Lambda_\varepsilon}[a_i, B_\mu]$  does *not* contain a term linear in  $a_i$  that otherwise would make the renormalized background perturbation theory ill-defined. Then, Eq. (1.2) yields a unique way to fix the remaining  $a_i$ -independent part of the above difference. At least when Eq. (1.3) is truncated to the leading  $L = 0$  order, the resulting  $\tilde{S}_{\Lambda_\varepsilon}[a_i, B_\mu]$  complies with the renormalizability from the power counting viewpoint, provided a pair of auxiliary ghost-fields is introduced.

Finally, once the residual symmetry (1.5) is fixed, thus implemented transformation (1.2) should yield such realization of the multi-scale decomposition of the theory (1.1) that the effective

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<sup>3</sup>This approximation is conventionally obtained from the renormalized representation  $\tilde{S}_{\Lambda_\varepsilon}^r[a_i, B_\mu]$  of  $\tilde{S}_{\Lambda_\varepsilon}[a_i, B_\mu]$  after the exclusion of the relevant counterterms.

action (1.4) is indeed of the Wilsonian type. Qualitatively, the action  $\tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu]$  should enforce that the Fourier images of  $a_\mu$  and  $B_\mu$  are dynamically localized in (but generically *not* limited to) the corresponding momentum slices  $[\Lambda_{int}, \Lambda_\varepsilon]$  and  $[0, \Lambda_{int}]$ . For this purpose, the interpolation scale  $\Lambda_{int}$  is to be identified with the *IR* limit  $\mathcal{M}_a$  (see eq. (6.1)) of the renormalized mass of the  $a_i$ -field in the auxiliary high-energy theory defining the correlators  $\langle \dots \rangle_a^B$ , while the limit  $\ln(\Lambda_\varepsilon/\Lambda_{int}) \sim 1/\varepsilon \rightarrow \infty$  is maintained via the dimensional regularization with  $\varepsilon = 4 - D \rightarrow +0$ . In due course, we demonstrate that the proposed below action  $\tilde{S}_{\Lambda_\varepsilon}[\cdot]$ , being conventionally renormalizable, satisfies certain precise conditions (5.3) which do imply the Wilsonian type of  $S_{eff}[B_\mu]$ . Also, to make the proposed perturbative computation of the effective action (1.2) tractable, it is important to choose such renormalization scale  $\Lambda = \tilde{\mathcal{M}}$  (implicitly entering, via  $g_r^2(\Lambda/\Lambda_{YM})$ , the definition of  $\langle a_\mu(\mathbf{x}) \rangle_a^B$  in eq. (1.3)) that is judiciously adjusted to the interpolation scale  $\Lambda_{int} = \mathcal{M}_a$  according to eq. (6.2). Altogether, thus implemented eq. (1.2) generalizes the transformation<sup>4</sup> [1] that, after infinitely many applications of its  $(\Lambda_\varepsilon - \Lambda_{int})/\Lambda_\varepsilon \rightarrow +0$  version for a fixed  $\Lambda_\varepsilon$ , facilitates modern approaches to perform the renormgroup reduction of the high-momentum Fourier modes of a given quantum field.

In Section 2, we introduce the relevant variety of the transformations (1.2) parameterized by a single function  $\mathcal{T}_i(B)$ . It is done in the simplest setting when the axial gauge  $A_\mu n_\mu = A_0 = 0$  is fixed prior to the transformation. It results in the theory of the two fields  $a_\mu$  and  $B_\mu$  where, in addition to the condition (1.6), the residual invariance (1.5) is also fixed by the second gauge condition  $B_\mu n_\mu = 0$ . In Section 3, synthesizing the latter variant of the transformation with a gauge fixing unity, we generalize the construction so that, keeping the symmetry (1.5) manifest, the decomposition is performed for a general class of gauge conditions for  $a_\mu$ . Also, we comment on the case when the auxiliary action  $\tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu]$  maintains a generic  $a_\mu$ -independent difference  $\tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu] - S_{YM}[a_\mu + B_\mu]$  so that the transformation (1.2) reduces to an identity (attributed to 't Hooft) used in an approach [6].

In Section 4, the condition (1.3) is reformulated as a simple algebraic equation that can be used to unambiguously determine the function  $\mathcal{T}_i(B)$  order by order in the framework of the renormalized loop-wise expansion applied prior to the averaging over  $B_\mu$ . The explicit form of  $\mathcal{T}_i(B)$  is obtained in the tree-order of this expansion, while the renormalizability of the resulting theory is sketched in Section 5, where the structure of the counterterms is also discussed. In Section 6, thus implemented transformation (1.2) is shown to guarantee the Wilsonian type of the effective action (1.4).

## 2 The general trick in the gauge $a_0 = B_0 = 0$

The short-cut route to a transformation (1.2) consistent with the symmetry (1.5) is to implement the transformation after the gauge fixing  $A_0 = 0$  so that the resulting action  $\tilde{S}_{\Lambda_\varepsilon}[a_i, B_i]$  forms the  $B_0 = 0$  reduction of a functional invariant under (1.5). The proposal is to first find such an action

$$S_m[a_i, B_\mu] = \left( \tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu] - S_{YM}[a_\mu + B_\mu] \right) \Big|_{a_0=0} \quad (2.1)$$

which resolves the condition that

$$1 = \int \mathcal{D}B_i \exp(-S_m[A_i - B_i, B_\mu]) \Big|_{B_0=A_0}, \quad (2.2)$$

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<sup>4</sup>Utilizing this transformation only once after a gauge fixing in the theory (1.1), one obtains the action  $\tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu]$  which differs from  $S_{YM}[a_\mu + B_\mu]$  only by kinetic terms quadratic in  $a_\mu$  and  $B_\mu$ . In turn, it allows to fulfil neither the symmetry (1.5) nor (even the tree-order approximation) the condition (1.3).

is fulfilled for an *arbitrary*  $A_\mu(\mathbf{x}) = (A_0(\mathbf{x}), A_i(\mathbf{x}))$ . Then, one is to insert  $B_0 = A_0 = 0$  option of the the unity (2.2) under the axial gauge  $A_0 = 0$  implementation of the generating functional

$$\langle \mathcal{Q}[A_\mu] \rangle_A = \frac{1}{Z_{YM}} \int \frac{\mathcal{D}A_\mu}{\mathcal{D}\omega} \exp(-S_{YM}[A_\mu]) \mathcal{Q}[A_\mu] , \quad (2.3)$$

where  $S_{YM}[A_\mu]$  is given in eq. (1.1),  $\mathcal{Q}[A_\mu]$  parameterizes a generic gauge-invariant external source,  $Z_{YM}$  denotes the partition function of the Euclidean gauge theory (2.3), and the measure  $\mathcal{D}A_\mu/\mathcal{D}\omega$  includes the normalization factor to cancel the volume  $\int \mathcal{D}\omega$  of the group of the standard gauge transformations. The reformulation (1.2) is completed through the subsequent change of the pair of the variables  $\mathcal{D}B_i \mathcal{D}A_i \rightarrow \mathcal{D}B_i \mathcal{D}a_i$ ,  $a_i + B_i = A_i$ . Altogether, it results in such decomposition<sup>5</sup> (1.2) where the integration over the high-energy modes  $a_\mu$  is performed in compliance with the  $B_0 = 0$  variant of the  $a_\mu n_\mu = a_0$  prescription:

$$\langle \mathcal{Q}[a_\mu + B_\mu] \rangle_a^{B^{ext}} = \frac{1}{Z_1[B]} \int \mathcal{D}a_\mu \delta(n_\mu a_\mu) \mathcal{Q}[a_\mu + B_\mu] \exp(-\tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu]) , \quad (2.4)$$

and the relevant microscopic action  $\tilde{S}_{\Lambda_\varepsilon}[a_i, B_\mu]$  is given by eq. (2.1), while the intermediate partition function (1.4) is such that  $\langle 1 \rangle_a^{B^{ext}} = 1$ .

Finally, it is straightforward to maintain that, in addition to (1.6), the above decomposition indeed implies one more axial gauge fixing associated with the transformations (1.5). For this purpose, it is sufficient to impose that  $S_m[a_i, B_\mu]$  is invariant under (1.5),

$$S_m[a_i, B_\mu] = S_m[a_i^{(\psi)}, B_\mu^{(\psi)}] \implies \tilde{S}_{\Lambda_\varepsilon}[a_i, B_\mu] = \tilde{S}_{\Lambda_\varepsilon}[a_i^{(\psi)}, B_\mu^{(\psi)}] , \quad (2.5)$$

which entails, due to eq. (2.1), the same invariance of the full action defined  $\tilde{S}_{\Lambda_\varepsilon}[\cdot]$ . Then, the remaining averaging over the low-energy modes can be reformulated as the  $B_0 = 0$  gauge implementation of the prescription which, similarly to eq. (2.3), manifestly respects gauge symmetry:

$$\langle \mathcal{G}[B_\mu] \rangle_B = \frac{1}{Z_{YM}} \int \frac{\mathcal{D}B_\mu}{\mathcal{D}\psi} \mathcal{G}[B_\mu] e^{-S_{eff}[B_\mu]} , \quad S_{eff}[B_\mu] = S_{eff}[B_\mu^{(\omega)}] \quad (2.6)$$

where  $Z_{YM}$  is the same as in (2.3),  $\langle 1 \rangle_B = 1$ , and the effective action (1.4) is gauge-invariant.

## 2.1 The ansatz for $S_m[a_i, B_\mu]$

To resolve the constraint (1.3) in the framework of the renormalized background perturbation theory, we propose to resolve the condition (2.2) by the  $\mathcal{T}_i(B)$ -dependent ansatz

$$S_m[a_i, B_\mu] = \mathcal{X}[w_j(a_i, B_\mu)] - \ln \left( \det \left[ \hat{\mathcal{E}}_{ij}(B) \right] \right) + \ln(\mathcal{Z}_\mathcal{X}) , \quad (2.7)$$

where  $B_\mu$  denotes the *full*  $D$ -vector, and

$$\mathcal{X}[w_i] = \int d^D x \frac{\mathcal{M}^2}{2g_r^2} \text{Tr} \left( w_i^2 \right) , \quad w_i(a_j, B_\mu) = -a_i - \frac{g_r^2}{\mathcal{M}^2} \mathcal{T}_i(B_\mu) , \quad (2.8)$$

and  $g_r^2 \equiv g_r^2(\Lambda/\Lambda_{YM})$  denotes the renormalized coupling constant  $g^2/Z_{g^2}$  in the original formulation (1.1) of the theory which is associated with a finite, when  $4 - D \rightarrow +0$ , normalization point

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<sup>5</sup>It reduces to the one of [1] provided  $2g_r^2 S_m[a_i, B_i] = \int d^D x \text{Tr} (a_i \hat{\mathcal{K}}_1^{ij} a_j + B_i \hat{\mathcal{K}}_2^{ij} B_j - (a_i + B_i) \hat{\mathcal{K}}^{ij} (a_j + B_j))$ , where  $\hat{\mathcal{K}}^{ij}$  is defined after eq. (5.1) and  $\hat{\mathcal{K}}_k^{ij}$  are generic operators satisfying the condition  $\hat{\mathcal{K}}^{-1} = \hat{\mathcal{K}}_1^{-1} + \hat{\mathcal{K}}_2^{-1}$ .

$\Lambda$ . Also, it is convenient to choose such value  $\check{\mathcal{M}}$  of  $\Lambda$  that the interpolation scale  $\Lambda_{int} = \mathcal{M}_a$  coincides with the parameter  $\mathcal{M}$  (as it will be formalized by eq. (6.2)).

As for  $\det[\cdot]$ , being evaluated with respect to both pairs of the indices of  $\hat{\mathcal{E}}_{ij}^{cd}[B]$ , it yields the Jacobian associated with the change of the variables  $B_i \rightarrow w_i(A_j - B_j, B_\mu)$  performed for a fixed  $B_0$ . The corresponding tensor-like operator  $\hat{\mathcal{E}}_{\mu\nu}^{cd}(B)$  is therefore defined via the relation

$$\langle \mathbf{y} | \left( \hat{\mathcal{E}}_{ij}^{cd}(B) - \delta_{ij} \delta^{cd} \right) | \mathbf{x} \rangle = - \frac{g_r^2}{\mathcal{M}^2} \frac{\delta \mathcal{T}_i^c(B(\mathbf{y}))}{\delta B_j^d(\mathbf{x})} , \quad (2.9)$$

and  $\mathcal{T}_\mu(\cdot)$  should depend only on the multiplicatively renormalized quantities which are finite in the limit  $\varepsilon = 4 - D \rightarrow +0$ . Finally, the constant  $\mathcal{Z}_{\mathcal{X}}$  is defined by the relation  $\mathcal{Z}_{\mathcal{X}} = \int \mathcal{D}w_i e^{-\mathcal{X}^r[w_i]}$  which ensures that, after the above change of the variables, the ansatz (2.7) indeed resolves the condition<sup>6</sup> (2.2). In turn, the required invariance (2.5) is evidently maintained provided

$$\mathcal{T}_j(B^{(\psi)}) = U(\psi) \mathcal{T}_j(B) U^{-1}(\psi) , \quad (2.10)$$

i.e., both the function  $\mathcal{T}_j(B(\mathbf{z}))$  and, in consequence, the operator  $\hat{\mathcal{E}}_{ij}(B)$  are transformed *covariantly* under the ordinary gauge symmetry.

Next, the high-energy averages (2.4) are, by construction, invariant under the replacement of  $\tilde{S}_{\Lambda_\varepsilon}^r[a_i, B_\mu]$  by the simpler action (2.11).

$$\bar{S}^r[a_i, B_\mu] = \check{S}^r[a_i, B_\mu] + \int d^D x \text{Tr}(a_j \mathcal{T}_j(B)) \quad (2.11)$$

resulting when the last two  $a_i$ -*independent* terms in eq. (2.7) are omitted so that

$$\check{S}^r[a_i, B_\mu] = S_{YM}^r[a_\mu + B_\mu] \Big|_{a_0=0} + \int d^D x \mathcal{M}^2 \text{Tr}(a_i^2) / 2g_r^2 , \quad (2.12)$$

where  $S_{YM}^r[A_\mu]$  is obtained from  $S_{YM}[A_\mu]$  rewriting  $g^2 = Z_{g^2} g_r^2$ . We utilize that the auxiliary theory (2.11), considered for a fixed  $B_\mu$  the axial gauge fixing for  $a_\mu$ , is conventionally renormalizable. It is also noteworthy that, according to the conditions (5.3), in the limit  $\varepsilon \rightarrow +0$  neither the parameter  $\mathcal{M}$  nor the involved gauge fields require a multiplicative renormalization in the framework of the background perturbation theory:  $a_i^r = a_i$ ,  $B_\mu^r = B_\mu$ ,  $\mathcal{M}^r = \mathcal{M}$ . The conditions (5.3) also imply that the renormalization of the entire gauge system *reduces* to the one of the perturbative expansion in the theory (2.11) of the single dynamical field  $a_i$ . In particular, for a given normalization point  $\Lambda$ , the renormalization of  $g^2$  in the latter theory is maintained via the same factor  $Z_{g^2}$  as in the original formulation (1.1) considered in the gauge  $A_0 = 0$ .

Finally, for our later purposes, we introduce vector-like ghost fields  $\bar{\vartheta}_i$  and  $\vartheta_i$  according to the representation

$$\det[\hat{\mathcal{K}}_{ij}(B)] = \int \mathcal{D}\bar{\vartheta}_i(\mathbf{z}) \mathcal{D}\vartheta_i(\mathbf{z}) \exp \left[ \int d^D x \text{Tr}(\bar{\vartheta}_i \hat{\mathcal{K}}_{ij}(B) \vartheta_j) \right] . \quad (2.13)$$

which, in effect, replaces  $\ln(\det[\cdot])$  in eq. (2.7) by the functional in the exponent in the r.h. side of eq. (2.13) so that  $\det[\hat{1}_{ij}] = 1$ . Let us stress that the above massive fermionic fields  $\bar{\vartheta}_i$ ,  $\vartheta_i$  should *not* be interpreted as some extra high-energy modes additional to  $a_i$ . Indeed, as the high-energy averages (2.4) are defined by the action (2.11), the second term in the r.h. side of eq. (2.7) enters the decomposition (1.2) only as the associated part of the low-energy effective action  $S_{eff}[B_\mu]$  entering the average (2.6).

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<sup>6</sup>In eq. (2.2), the functional measure is presumed to be defined so that  $\mathcal{Z}_{\mathcal{X}}$  is finite when  $4 - D \rightarrow +0$ .

### 3 Restoration of the explicit background gauge invariance

Actually, the condition (2.2) can be generalized to implement a generic gauge fixing for the field  $a_\mu$  keeping the background gauge invariance (1.5) manifest. Given the generalized construction (see eq. (3.1) below), eq. (2.2) is reproduced imposing in  $\mathbf{E}^4$  the double axial gauge  $n_\mu a_\mu^b = n_\mu B_\mu^b = 0$  in the two successive steps so that the invariance (1.5) is fixed only in the very end. For simplicity, we restrict our attention to the subvariety of the *linear* background gauges  $f^c(a, B) = \mathcal{R}_\mu^{ce}(B) a_\mu^e = 0$  where, in order to maintain the required symmetry of  $\tilde{S}_{\Lambda_\varepsilon}[a, B]$ , the operator  $\mathcal{R}_\mu(B)$  is constrained to transform homogeneously under the transformations (1.5):  $\mathcal{R}_\mu(B^{(\psi)}) = U(\psi)\mathcal{R}_\mu(B)U^{-1}(\psi)$ .

The form of the multi-scale decomposition, respecting the latter symmetry, can be introduced judiciously synthesizing a transformation like (1.2) with the Faddeev-Popov unity adapted to fix, in accordance with (1.5), a gauge for the high-energy field  $a_\mu$  represented by the combination  $A_\mu - B_\mu$ . The proposal is to utilize the following  $B_\mu$ -dependent functional

$$1 = \int \frac{\mathcal{D}B_\mu}{\mathcal{D}\psi} \exp\left(-\tilde{S}_m[A_\mu^{(\omega_0[B])} - B_\mu, B_\mu]\right) \quad (3.1)$$

as the composed unity, where  $A_\mu^{(\omega)} = U(\omega)(A_\mu + i\partial_\mu)U^{-1}(\omega)$  and the auxiliary action  $\tilde{S}_m[a_\mu, B_\mu]$ , being invariant under (1.5), is such that the condition (3.1) holds true for  $\forall A_\mu$ . Implying the necessity of the factor  $1/\mathcal{D}\psi$ , the functional  $\omega_0[B] \equiv \omega_0[A_\mu, B_\mu]$  is determined by the relation

$$e^{-\tilde{S}_m[A_\mu^{(\omega_0[B])} - B_\mu, B_\mu]} = \int \mathcal{D}\omega \det\left[f'(A_\mu^{(\omega)} - B_\mu, B_\mu)\right] \delta\left(f(A_\mu^{(\omega)} - B_\mu, B_\mu)\right) e^{-\tilde{S}_m[A_\mu^{(\omega)} - B_\mu, B_\mu]} \quad (3.2)$$

where  $f'(\cdot) \equiv \delta f(\cdot)/\delta\omega$ , and the shift  $\omega \rightarrow \omega \circ \psi$  reveals that the r.h. side is invariant under the  $a_\mu \rightarrow A_\mu^{(\omega)} - B_\mu$  option of the transformations (1.5) with  $A_\mu^{(\omega \circ \psi)} - B_\mu^{(\psi)} = U(\psi)(A_\mu^{(\omega)} - B_\mu)U^{-1}(\psi)$ . In turn, it implies that  $\omega_0[A_\mu, B_\mu^{(\psi)}] = \omega_0[A_\mu, B_\mu] \circ \psi$  which explains the necessity to cancel in eq. (3.1) the volume  $\int \mathcal{D}\psi$  of the group of the transformations (1.5).

Then, akin to the previous Section, one is to insert the unity (3.1) under the functional integral (2.3) and, after simple manipulations, we arrive at the relation (1.2). Its particular form is specified by eqs. (2.4) and (2.6), provided the identification

$$S_m[a_\mu, B_\mu] = \tilde{S}_m[a_\mu, B_\mu] - \ln(\det[\mathcal{R}_\mu(B)D_\mu(a + B)]) \quad (3.3)$$

is made in the definition (2.1) of  $\tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu]$ , while  $n_\mu$  is replaced by  $\mathcal{R}_\mu(B)$  in eq. (2.4). In consequence, employing that  $\mathcal{R}_\mu(B^{(\psi)}) = U(\psi)\mathcal{R}_\mu(B)U^{-1}(\psi)$ , the condition (2.5) is indeed sufficient to maintain the background gauge invariance (1.5) of thus introduced action  $\tilde{S}_{\Lambda_\varepsilon}[a_\mu, B_\mu]$ . In turn, it allows to rewrite the condition (3.1) in the form:

$$1 = \int \mathcal{D}B_\mu \det[\mathcal{R}_\mu(B)D_\mu(A)] \delta(\mathcal{R}_\mu(B)(A_\mu - B_\mu)) \exp\left(-\tilde{S}_m[A_\mu - B_\mu, B_\mu]\right) \quad (3.4)$$

In the axial gauge (1.6), integrating over the longitudinal component  $n_\mu B_\mu$  of  $B_\mu$ , one reduces the condition (3.4) to the constraint (2.2).

Finally, we remark that, when  $\tilde{S}_m[a_\mu, B_\mu] = \tilde{S}_m[B_\mu]$  is  $a_\mu$ -independent, the insertion of the unity (3.1) does not impose any gauge fixing for  $a_\mu = A_\mu - B_\mu$  which can be performed subsequently. Thus reduced unity (3.1) yields the transformation of the generation functional (2.3) which reproduces the so-called 't Hooft identity that, in [6], is claimed (without a specification of  $\tilde{S}_m[B_\mu]$ ) to help in separation of confining  $B_\mu$ -configurations. Irrespectively of a choice of

$\tilde{S}_m[B_\mu]$ , such a transformation is ineffective to implement a multi-scale decomposition: it does *not* attribute a mass term to the field  $a_\mu$ . Consequently, prior to the integration over  $B_\mu$ , the contribution of the low-energy modes of  $a_\mu$  is unsuppressed, and the effective action (1.4) is not of the Wilsonian type. Also, for  $\forall \tilde{S}_m[B_\mu]$ , the condition (1.3) is violated already at the tree-level of the loop-wise expansion.

## 4 Resolving the constraint (1.3)

To demonstrate that the condition (1.3) unambiguously determines the function  $\mathcal{T}_\mu(B)$  entering the ansatz (2.8), we begin with the following observation. To begin with, presuming  $n_\mu a_\mu = a_0$ , eq. (1.3) can be rewritten as the constraint  $\delta \mathcal{W}^r[J_i|B_\mu]/\delta J_j(\mathbf{z})|_{J_i=0} = 0$ . Here,  $\mathcal{W}^r[\cdot]$  denotes the relevant renormalized generating functional expressed in terms of the coupling constant  $g_r^2 = g^2/Z_{g^2}$  and the fields  $a_i = a_i^r$ ,  $B_\mu = B_\mu^r$  renormalized according to the discussion after eq. (2.12). This functional is defined by the relation  $e^{\mathcal{W}^r[J_i|B_\mu]} = \langle e^{\int d^D x \text{Tr}(a_i J_i)} \rangle_a^{B^{ext}}$ , where the high-energy quantum averaging is introduced in eq. (2.4). In turn, the latter constraint can be replaced by the condition

$$\delta \Gamma^r[C_i|B_\mu]/\delta C_j(\mathbf{z})\big|_{C_i=0} = 0 \quad (4.1)$$

formulated in terms of the *renormalized* Legendre effective action  $\Gamma^r[C_i|B_\mu]$  determined by the canonical relation:  $\Gamma^r[C_i|B_\mu] + \mathcal{W}^r[J_i|B_\mu] - \int d^D x \text{Tr}(J_i C_i) = 0$ , where  $J_i^r = J_i$  and  $C_i^r = C_i$  in view of eq. (5.4). Indeed, eq. (4.1) follows from the general property that  $\delta \Gamma^r[C_i|B_\mu]/\delta C_j(\mathbf{z}) = 0$  for  $C_i(\mathbf{x}) = \langle a_i(\mathbf{x}) \rangle_a^{B^{ext}}$  which is a consequence of the above relation between  $\Gamma^r[\cdot]$  and  $\mathcal{W}^r[\cdot]$ .

Next, we utilize that the constraint (4.1) is not changed if  $\Gamma^r[C_i|B_\mu]$  is replaced by the Legendre effective action in the simpler theory with the modified action (2.11). Furthermore, using the *linearity* of the reduced action (2.11) in  $\mathcal{T}_j(B)$ , one readily obtains that  $\Gamma^r[C_i|B_\mu] = \check{\Gamma}^r[C_i|B_\mu] + \int d^D x \text{Tr}(C_j \mathcal{T}_j(B))$ , where  $\check{\Gamma}^r[C_i|B_\mu]$  is the renormalized Legendre effective action associated with the generating functional defined by the action (2.12). Altogether, implementing the background perturbation theory associated with a given normalization point  $\Lambda$ , eq. (4.1) can be rewritten in the form

$$\mathcal{T}_j(B(\mathbf{z})) = -\frac{\delta \check{\Gamma}^r[C_i|B_\nu]}{\delta C_j(\mathbf{z})}\bigg|_{C_i=0} = \sum_{l=0}^{\infty} \mathcal{T}_j^{(l)}(B) , \quad (4.2)$$

where the expansion of  $\mathcal{T}_j(B)$  is generated by the renormalized loop-wise expansion of  $\Gamma^r[\cdot|\cdot] = \sum_{l=0}^{\infty} [\check{\Gamma}_l^r[\cdot|\cdot] + \int d^D x \text{Tr}(C_j \mathcal{T}_j^{(l)}(\cdot))]$  so that  $\mathcal{T}_j^{(l)}(B) \sim g_r^{2l-2}(\Lambda/\Lambda_{YM})$ . To complete the specification of the renormalized form of the action (3.3), the (truncated) pattern (4.2) is to be substituted into eq. (2.9) defining the operator  $\hat{\mathcal{E}}_{ij}(B) = \sum_l \hat{\mathcal{E}}_{ij}^{(l)}(B)$ .

In the tree-approximation, the ansatz (2.7) is defined by  $\check{\Gamma}_0^r[C_i|B_\nu] = \check{S}^{tr}[C_i, B_\nu]$ :

$$\left(\mathcal{T}_j^{(0)}(B(\mathbf{z}))\right)^b = -\frac{\delta \check{S}^{tr}[C_i, B_\nu]}{\delta C_j^b(\mathbf{z})}\bigg|_{C_i=0} = -\frac{\delta S_{YM}^{tr}[B_\nu]}{\delta B_j^b(\mathbf{z})} = \frac{1}{g_r^2} D_\mu^{bc}(B(\mathbf{z})) F_{\mu j}^c(B(\mathbf{z})) , \quad (4.3)$$

where the tree-level action  $\check{S}^{tr}[\cdot]$  is conventionally obtained from eq. (2.12) replacing  $S_{YM}^r[B_\nu]$  by  $S_{YM}^{tr}[B_\nu]$  which, in turn, implies the replacement of  $g^2 = Z_{g^2} g_r^2$  by  $g_r^2 \equiv g_r^2(\Lambda/\Lambda_{YM})$ . Correspondingly, the leading approximation to operator  $\hat{\mathcal{E}}_{ij}(B)$  reads

$$\left(\hat{\mathcal{E}}_{ij}^{(0)}(B)\right)^{bd} - \delta_{ij} \delta^{bd} = -\frac{1}{\mathcal{M}^2} \left( D_\rho^{bc}(B) D_\rho^{cd}(B) \delta_{ij} - D_i^{bc}(B) D_j^{cd}(B) - 2 f^{bde} F_{ij}^e(B) \right) . \quad (4.4)$$



Next, to evaluate the average (2.4) up to a given order  $L \geq 0$  of the loop-wise expansion,  $\check{\Gamma}^r[C_i|B_\nu]$  is sufficient to determine up to the same order of the expansion. In eq. (2.11), in the sum  $\sum_{l=0}^L a_j \mathcal{T}_j^{(l)}(B)$ , only the term  $a_j \mathcal{T}_j^{(0)}(B)$  is to be involved in the derivation of the propagator  $G_{ij}^r(\mathbf{y}, \mathbf{x}|B)$  of the renormalized perturbation theory for a fixed  $B_\mu$ . Then, according to eq. (4.1), (for  $\forall L$ ) the tree-level approximation  $\bar{S}^{tr}[a_i, B_\nu]$  to the associated renormalized action (2.11) has *vanishing* linear in  $a_\mu$  term,  $\delta \bar{S}^{tr}[a_i, B_\nu]/\delta a_i|_{a_i=0} = 0$  for *any*  $B_\mu$ , that is necessary for self-consistency of the background perturbation theory. Correspondingly, the propagator reads  $G_{ij}^r(\mathbf{y}, \mathbf{x}|B) = g_r^2 \mathcal{M}^{-2} < \mathbf{y} | (\hat{\mathcal{E}}_{ij}^{(0)}(B))^{-1} | \mathbf{x} >$ , where  $\hat{\mathcal{E}}_{ij}^{(0)}(B)$  is given by eq. (4.4). As for the difference between the relevant action (2.11) and  $\mathcal{M}^2 \int d^D x \text{Tr}(a_i \hat{\mathcal{E}}_{ij}^{(0)}(B) a_j)/2g_r^2$ , for a given  $L$ , it assumes the form of the sum of the counterterms (5.2) (truncated up to the  $L$ th order of the expansion) and the remaining part  $\int d^D x \text{Tr}(-(4i[a_q, a_l] D_q(B) a_l + [a_q, a_l][a_q, a_l])/4g_r^2 + \sum_{l=1}^L a_j \mathcal{T}_j^{(l)}(B))$  of  $\bar{S}^{tr}[\cdot]$ , with the  $l \geq 1$  terms  $\text{Tr}(a_j \mathcal{T}_j^{(l)}(B))$  being treated as additional vertices.

By virtue of eq. (4.1), for a given  $l$ , the role of the latter term is to exactly cancel, for  $\forall B_\mu$ , the 1PI tad-pole-like subgraphs which are associated with the  $l$ th order of the loop-wise expansion of  $< a_i >_a^B$  evaluated in the theory (2.12). When the condition (1.3) is violated already in the classical limit, the self-consistency of the weak-coupling series (developed for correlators (2.4)) is spoiled by the proliferation of the tree-like subgraphs. Being generated by the diagrammatic expansion of the  $g_r^0$ th contribution to  $< a_i >_a^B$  in the theory (2.11), they are attached to the rest of a graph by a single 'external'  $a_\mu$ -line. Proliferation of these subgraphs is not suppressed: once  $\mathcal{T}_j^{(0)}(B) \sim g_r^{-2}$ , they are *all* of the same  $g_r^0$ th order. But, once eq. (1.3) holds true classically, the proliferation of the associated with  $< a_i >_a^B$  subgraphs is already suppressed by powers of  $g_r^2$ .

Finally, the relation (4.2) is consistent with the condition (2.10). The consistency is maintained provided  $\check{\Gamma}^r[C_i|B_\mu] = \check{\Gamma}^r[C_i^{(\psi)}|B_\mu^{(\psi)}]$  is invariant under the  $a_i \rightarrow C_i$  option of the transformations (1.5) once eq. (2.10) (and, in consequence, eq. (2.5)) is satisfied. In turn, this invariance follows from the observation that the symmetry (2.5) guarantees the invariance  $\mathcal{W}^r[a_i^{(\psi)}|B_\mu^{(\psi)}] = \mathcal{W}^r[a_i|B_\mu]$  of the associated generating functional  $\mathcal{W}^r[J_i|B_\mu]$  under the transformations (1.5).

## 5 Renormalizability of the novel representation

Employing the representation (2.13), we are ready to prove that the ansatz (2.7), implemented in the tree-order approximation (4.3), results in the action  $\tilde{S}_{\Lambda_\varepsilon}[a_i, B_\mu|\bar{\vartheta}_i, \vartheta_i]$  defining the theory *renormalizable* from the power counting viewpoint. It is most transparent in the gauge  $a_0 = B_0 = 0$  presumed till the end of the paper. In this case, for a given normalization point  $\Lambda$ , the quadratic in  $a_i$  and  $B_i$  part of the tree-level approximation  $\tilde{S}_{\Lambda_\varepsilon}^{tr}[\cdot]$  (to the renormalized action  $\tilde{S}_{\Lambda_\varepsilon}^r[\cdot]$  defined by eq. (2.1)) assumes the form  $\int d^D x \text{Tr}(a_i \hat{\mathcal{K}}_1^{ij} a_j + B_i \hat{\mathcal{K}}_2^{ij} B_j)/2g_r^2(\Lambda/\Lambda_{YM})$ , where

$$\hat{\mathcal{K}}_1^{ij} = \hat{\mathcal{K}}^{ij} + \mathcal{M}^2 \delta^{ij} \quad , \quad \hat{\mathcal{K}}_2^{ij} = \hat{\mathcal{K}}^{ij} (1 + \Delta/\mathcal{M}^2) \quad , \quad (\hat{\mathcal{K}}^{-1})^{ij} = \sum_{m=1}^2 (\hat{\mathcal{K}}_m^{-1})^{ij} \quad , \quad (5.1)$$

where  $\hat{\mathcal{K}}^{ij} = \hat{P}^{ij} \Delta - (\delta^{ij} - \hat{P}^{ij}) \partial_0^2$  is the operator defining (modulo the factor  $1/2g_r^2$ ) the quadratic part of the action (1.1), while  $\hat{P}^{ij} = \delta^{ij} - \partial^i \partial_l^{-2} \partial^j$ ,  $\Delta = -\partial_l^2$  and  $(\delta^{ij} - \hat{P}^{ij}) \hat{P}^{ij} = 0$ . In particular, eq. (5.1) implies that, despite the presence of the mass-term, among the three components of the field  $a_i$  there are only *two* propagating polarizations selected by the projector  $\hat{P}^{ij}$ .

Next, while the dimensions of the fields are  $[a_i] = [\bar{\vartheta}_i] = [\vartheta_i] = 1$  and  $[B_i] = 0$ , thus implemented action  $\tilde{S}_{\Lambda_\varepsilon}[a_i, B_i]$  generates *no* vertices with a positive dimension so that there is only a finite number of correlation functions comprised of superficially divergent 1PI graphs.

Altogether, as it will be sketched in the end of Section 6, the pattern of the counterterms reads

$$\tilde{S}_{CT}[a_i, B_i|\bar{\vartheta}_i, \vartheta_i] = \frac{(Z_{g^2}^{-1} - 1)}{4g_r^2} \int d^D x \text{Tr} (F_{\mu\nu}(a_i + B_i))^2 . \quad (5.2)$$

where  $g_r = g_r(\Lambda/\Lambda_{YM})$ , and the vector-like ghosts  $\bar{\vartheta}_i, \vartheta_i$  are treated as independent dynamical fields explicitly involved in the renormalization algorithm. It is crucial that the factor  $Z_{g^2}$ , being the same as in the standard formulation (1.1) considered in the gauge  $A_0 = 0$ , is accumulated by the divergent perturbative diagrams *without* internal lines associated either with the low-energy field  $B_i$  or with the latter ghosts. In consequence, the counterterms comply with the condition

$$\tilde{S}_{CT}[a_i, B_i|\bar{\vartheta}_i, \vartheta_i] = \bar{S}_{CT}^{(pt)}[a_i, B_i] \quad , \quad \tilde{S}_{CT}[a_i, B_i|\bar{\vartheta}_i, \vartheta_i] = S_{CT}[a_i + B_i] , \quad (5.3)$$

where  $\bar{S}_{CT}^{(pt)}[a_i, B_i]$  stands for the counterterms relevant for the background perturbation theory (i.e., prior to the integration over  $B_i$ ) applied to the averages  $\langle .. \rangle_a^B$  in the theory (2.11). The second part of eq. (5.3) states that the replacement  $a_i + B_i \rightarrow A_i$  transforms the r.h. side of eq. (5.2) into the well-known pattern of the counterterms  $S_{CT}[A_i]$  evaluated in the framework of the original representation (1.1) in the gauge  $A_0 = 0$ .

Eq. (5.3) implies in particular that, given the double axial gauge fixing  $a_0 = B_0 = 0$ , the non-renormalization of the involved gauge fields is valid not only in the background perturbation theory (for a fixed  $B_i$ ) but also in the full theory (of the two dynamical fields  $a_i$  and  $B_i$ ):

$$a_i^r = a_i \quad , \quad B_i^r = B_i \quad , \quad g^2 = Z_{g^2} g_r^2 . \quad (5.4)$$

The ghost-fields are not renormalized either:  $\bar{\vartheta}_i^r = \bar{\vartheta}_i$ ,  $\vartheta_i^r = \vartheta_i$ . Also, neither the "bare" mass  $M_a = \mathcal{M}$  of  $a_i$  nor the "bare" mass  $M_{gh} = \mathcal{M}$  of  $\bar{\vartheta}_i, \vartheta_i$  require any divergent (when  $\varepsilon \rightarrow +0$ ) multiplicative renormalization both prior and after the integration over  $B_i$ . In consequence, the part  $S_m[a_i, B_i|\bar{\vartheta}_i, \vartheta_i]$  of  $\tilde{S}_{\Lambda_\varepsilon}[a_i, B_i|\bar{\vartheta}_i, \vartheta_i]$ , resulting after the reformulation (2.13), contributes to the counterterms neither in the full theory nor in the background perturbation theory.

Finally, observe that eq. (5.1) displays the basic feature of the multiscale decomposition: the propagators  $\delta^{ce}\mathcal{D}_1^{ij}(\mathbf{p})$  and  $\delta^{ce}\mathcal{D}_2^{ij}(\mathbf{p})$  of  $a_i^c$  and  $B_i^c$  ( $\langle \mathbf{p} | (\hat{\mathcal{K}}_k^{ij})^{-1} | 0 \rangle = \mathcal{D}_k^{ij}(\mathbf{p})$ ) approach the propagator  $\delta^{ce}\mathcal{D}^{ij}(\mathbf{p}) = \delta^{ce} \langle \mathbf{p} | \hat{\mathcal{K}}_i^{-1} | 0 \rangle$  of the field  $A_i^c$  (of eq. (1.1)) in the *UV* and *IR* domains of the momentum squared respectively. Owing to the last relation of eq. (5.1), it implies that  $\mathcal{D}_1^{ij}(\mathbf{p}) \gg \mathcal{D}_2^{ij}(\mathbf{p})$  and  $\mathcal{D}_1^{ij}(\mathbf{p}) \ll \mathcal{D}_2^{ij}(\mathbf{p})$  for  $\mathbf{p}^2 \gg \mathcal{M}^2$  and  $\mathbf{p}^2 \ll \mathcal{M}^2$  correspondingly. E.g.,  $\mathcal{D}_1^{ij}(\mathbf{p}) \sim (\mathbf{p}^2)^0$  when  $\mathbf{p}^2 \rightarrow 0$ , while  $\mathcal{D}_2^{ij}(\mathbf{p}) \sim (\mathbf{p}^2)^{-2}$  for  $\mathbf{p}^2 \equiv p_\mu^2 \rightarrow \infty$ .

## 6 The effective action is of the Wilsonian type

Given the dimensional regularization  $4-D = \varepsilon \rightarrow +0$  and the gauge condition  $a_0 = B_0 = 0$ , let us first adapt the conventional requirement, maintaining that an effective action is of the Wilsonian type, to the specific case (1.4) corresponding to the ansatz (2.7) fixed by eq. (4.3). Secondly, we verify that the conditions (5.3) are sufficient to fulfill this requirement. To begin with,  $S_{eff}[B]$  should describe low-energy dynamics separated by a *finite* (for  $\varepsilon \rightarrow +0$ ) *UV* cut off  $\Lambda_{int}$ . In our case,  $\Lambda_{int}$  is naturally identified with the *IR* limit  $\mathcal{M}_a = \mathcal{M}/Z_{\mathcal{M}}^{(pt)}$  of the renormalized mass of the field  $a_i$  in the auxiliary high-energy theory (2.11) for a fixed  $B_i$ . Provided  $\mathcal{M}_a$  is finite when  $\varepsilon \rightarrow +0$  and employing the non-renormalization (5.4) of  $B_i = B_i^r$ , the requirement reads: the operator expansion of this action is expressed,  $S_{eff}[B] = \sum_{n \geq 1} c_n(\mathcal{M}_a) \mathcal{O}_n[B]$ , in terms of  $B_i$  and *renormalized* coupling constants  $c_n(\mathcal{M}_a)$ . The coefficients  $c_n(\mathcal{M}_a)$  are given by the  $\bar{\Lambda} = \mathcal{M}_a$

option of the "running" constants  $c_n(\bar{\Lambda})$  which, for any  $\varepsilon$ -independent  $\bar{\Lambda}$ , should possess a finite limit when  $\varepsilon \rightarrow +0$ . It means that the effective theory is free of  $UV$  divergences which are regularized due to an implicit  $UV$  cutoff of order of  $\mathcal{M}_a$  implemented by the action  $S_{eff}[B]$ .

In the auxiliary theories (2.11) and (2.12) considered for a fixed  $B_i$ , the  $IR$  limit  $\mathcal{M}_a \equiv \mathcal{M}_a(\mathcal{M}, \Lambda, \Lambda_{YM})$  of the renormalized  $a_i$ -mass is defined (see below) by the relation

$$\mathcal{M}_a/\tilde{g}_r(\mathcal{M}_a/\Lambda_{YM}) = \mathcal{M}/g_r(\Lambda/\Lambda_{YM}), \quad (6.1)$$

where  $\tilde{g}_r^2(\mathcal{M}_a/\Lambda_{YM})$  denotes the  $IR$  limit (to be introduced after eq. (6.3)) of the coupling constant in the latter auxiliary theories. Therefore, the scale  $\mathcal{M}_a = \Lambda_{int}$  of the interpolation is *finite* in the limit  $\varepsilon \rightarrow +0$  provided both  $\mathcal{M}$  and  $\Lambda$  are chosen to be  $\varepsilon$ -independent (in compliance with the conditions (5.3)). Once  $\mathcal{M}_a$  is finite, the above requirement on  $S_{eff}[B]$  is tantamount to the first of the conditions (5.3) imposed on the counterterms of the *microscopic* theory determined by the (conventionally renormalizable) action  $\tilde{S}_{\Lambda_\varepsilon}[a_i, B_i]$ . Indeed, it justifies that in the  $\varepsilon \rightarrow +0$  limit the effective theory is free of  $UV$  divergences. Correspondingly, the action  $S_{eff}[B] = \sum_{n \geq 1} c_n(\mathcal{M}_a) \mathcal{O}_n[B]$  is  $1/\varepsilon$ -independent when expressed in terms of  $\mathcal{M}_a$ ,  $B_i = B_i^r$  and  $g_r^2 = g^2/Z_{g^2}$ , where  $Z_{g^2}$  is defined by eq. (5.2).

Applying the renormalized background perturbation theory (combined with the covariant derivatives' expansion), the computation of the effective action (1.4) considerably simplifies when, in the theory (2.11), the  $IR$  limit  $\mathcal{M}_a = \mathcal{M}_a(\mathcal{M}, \Lambda, \Lambda_{YM})$  of the  $a_i$ -mass coincides with the tree-level approximation  $\mathcal{M}$  to this mass. To this aim, one is to select such  $\Lambda = \check{\mathcal{M}}$  that

$$g_r^2(\check{\mathcal{M}}/\Lambda_{YM}) = \tilde{g}_r^2(\mathcal{M}/\Lambda_{YM}) \quad \Longleftrightarrow \quad \mathcal{M}_a(\mathcal{M}, \check{\mathcal{M}}, \Lambda_{YM}) = \mathcal{M}. \quad (6.2)$$

Introducing the reparameterization  $\Lambda \rightarrow \tilde{\Lambda}(\Lambda) \equiv \tilde{\Lambda}(\Lambda, \Lambda_{YM})$  via the relation  $g_r^2(\Lambda/\Lambda_{YM}) = \tilde{g}_r^2(\tilde{\Lambda}/\Lambda_{YM})$ , we obtain  $\check{\mathcal{M}} = \tilde{\Lambda}^{-1}(\mathcal{M})$ . In accordance with the concept of the anomalous dimension, the relation (6.1) between  $\mathcal{M}_a$  and  $\mathcal{M}$  implies then that, for a fixed  $\mathcal{M}_a$  and  $\forall \Lambda \geq \check{\mathcal{M}} = \tilde{\Lambda}^{-1}(\mathcal{M}_a)$ , the quantity  $\mathcal{M} = \mathcal{M}(\Lambda)$  can be reinterpreted as the running mass associated with the scale  $\Lambda$  (with  $\mathcal{M}(\check{\mathcal{M}}) = \mathcal{M}_a$ ).

In conclusion, let us sketch the derivation of eqs. (5.3) and (6.1). To justify the second of the conditions (5.3), it is convenient to treat the action  $\tilde{S}_{\Lambda_\varepsilon}[a_i, B_\mu|\bar{\vartheta}_i, \vartheta_i]$ , defined by the trick (2.2)/(2.7) together with the specification (4.3) of  $\mathcal{T}_j(B)$ , as belonging to the *two*-parametric variety. For this purpose,  $S_m^r[\cdot]$  is generalized to  $S_m^r[a_i, B_\mu|\{\xi_k\}]$  so that, in eq. (2.8),  $w_\mu$  is replaced by  $\xi_1 a_\mu + \xi_2 g_r^2 \mathcal{T}_\mu(\cdot)/\mathcal{M}^2$ . As the considered implementation of the transformation (1.2) keeps intact the renormalizability in the power counting sense, there should exist such multiplicative renormalization both of  $\xi_k = Z_{\xi_k} \xi_k^r$  and of  $\bar{\vartheta}_i = Z_{\bar{\vartheta}} \bar{\vartheta}_i^r$ ,  $\vartheta_i = Z_{\vartheta} \vartheta_i^r$  that, together with eq. (5.4), allows to separate the relevant counterterms  $\tilde{S}_{CT}[a_i, B_\mu|\bar{\vartheta}_i, \vartheta_i]$  to cancel all the  $UV$  divergences in the theory with thus specified  $\tilde{S}_{\Lambda_\varepsilon}[\cdot|\cdot]$ . In view of the non-renormalization (5.4) of the fields (which holds true by virtue of the non-renormalization  $A_i = A_i^r$  in the axial gauge option of the original formulation (1.1)) the difference between the ghost-independent counterterms  $\tilde{S}_{CT}[a_i, B_i|0, 0]$  and  $S_{CT}[a_i + B_i]$  may be composed (by virtue of eq. (2.1) only of  $\beta_1 = \int d^D x \text{Tr}(a_i^2)$ ,  $\beta_2 = \int d^D x \text{Tr}(a_i \delta S_{YM}[B_k]/\delta B_i)$ , and  $\beta_3 = \int d^D x \text{Tr}((\delta S_{YM}[B_k]/\delta B_i)^2)$ ). In the two-parametric variety, to guarantee that  $\tilde{S}_{CT}[a_i, B_i|0, 0] = S_{CT}[a_i + B_i]$ , it is sufficient to verify the absence of the counterterms proportional to any two  $\beta_q$ . We choose  $q = 1, 3$  which, in particular, would imply that  $\delta M_a^2 = 0$ , where  $\delta M_a^2$  denotes the divergent renormalization of the squared mass  $M_a^2$  of  $a_i$ . On the other hand, the power-counting proves that a possible ghost-dependent part of  $\tilde{S}_{CT}[a_i, B_i|\bar{\vartheta}_i, \vartheta_i]$  may be associated only with the renormalization  $\delta M_{gh}^2$  of the mass  $\mathcal{M}$  of the ghosts which is excluded since the ansatz (2.7) guarantees that  $\delta M_{gh}^2 = \delta M_a^2 = 0$ .

Concerning the required verification, the power-counting demonstrates that, in the considered variety of the theories, the divergent perturbative diagrams do not generate the combination

$\beta_3$ . To justify that the remaining combination  $\beta_1 \equiv \beta_1[a]$  is not generated either, we make the inverse change  $\mathcal{D}B_i \mathcal{D}a_i \rightarrow \mathcal{D}B_i \mathcal{D}A_i$  of the variables to show that the difference  $\Delta \tilde{S}_{CT}[A_i, B_i] = \tilde{S}_{CT}[A_i - B_i, B_i|0, 0] - S_{CT}[A_i]$  may be only such functional that  $\Delta \tilde{S}_{CT}[A_i, 0] = 0$  for  $\forall A_i$  (which excludes  $\beta_1[A-B]$  since  $\beta_1[A-B]|_{B=0} \neq 0$  for  $A_i^2 \neq 0$ ). For this purpose, consider the generating functional  $\mathcal{W}[J_i^+, I_i]$  (with  $\mathcal{W}[0, 0] = 0$ ) which results after the averaging of  $e^{\int d^D x \text{Tr}(J_i^+ A_i + I_i B_i)}$  in the theory defined by the action  $\tilde{S}_{\Lambda_\varepsilon}[A_i - B_i, B_i]$  (implicitly depending on  $\{\xi_k\}$ ). Integration over  $B_i$  yields  $e^{\mathcal{W}[J_i^+, I_i]} = \langle e^{\int d^D x \text{Tr}(J_i^+ A_i)} \rangle_A^I$  where the  $A_i$ -averaging is performed with respect to the action  $(S_{YM}[A_i] - \Delta \tilde{\mathcal{W}}[I_i|A_i])$ , and  $e^{\Delta \tilde{\mathcal{W}}[I_i|A_i]} = \int \mathcal{D}B_i e^{-S_m[A_i - B_i, B_i|\{\xi_k\}] + \int d^D x \text{Tr}(I_i B_i)}$ . Eq. (2.2) leads to  $\Delta \tilde{\mathcal{W}}[0|A_i] = 0$  which, in turn, implies the required condition  $\Delta \tilde{S}_{CT}[A_i, 0] = 0$ . Indeed, the constraint  $\Delta \tilde{\mathcal{W}}[0|A_i]$  means that  $\Delta \tilde{S}_{CT}[A_i, B_i]$  may be generated only by those 1PI diagrams which necessarily possess a *nonzero* number of external  $B_i$ -lines. These lines are associated with subgraphs composed into correlation functions which are obtained applying the functional  $\delta/\delta I_i(\mathbf{x})$ -derivatives to the intermediate generating functional  $\Delta \tilde{\mathcal{W}}[I_i|A_i]$ .

To justify the first of the conditions (5.3), we prove that  $\tilde{S}_{CT}^{(pt)}[a_i, B_i] = S_{CT}[a_i + B_i]$  which, in view of the identity<sup>7</sup>  $\check{S}_{CT}^{(pt)}[a_i, B_i] = \tilde{S}_{CT}^{(pt)}[a_i, B_i]$ , is a consequence the condition  $\check{S}_{CT}^{(pt)}[a_i, B_i] = S_{CT}[a_i + B_i]$ , where  $\check{S}_{CT}^{(pt)}[\cdot]$  denotes the counterterms in the theory defined, for a fixed  $B_i$ , by the action (2.12). To verify the latter condition, let us temporarily omit the contribution associated with the mass term of the  $a_i$ -field. Then, it is easy to derive that the corresponding Legendre effective action (LEA)  $\check{\Gamma}[C_i|B_i]|_{\mathcal{M}=0} = \Gamma_{YM}[C_i + B_i]$ , where  $\Gamma_{YM}[C_i]$  denotes LEA in the theory (1.1) in the gauge  $A_0 = 0$ . In consequence,  $\check{S}_{CT}^{(pt)}[a_i, B_i]|_{\mathcal{M}=0} = S_{CT}[a_i + B_i]$ , and eq. (5.4) remains valid. Reintroducing the mass term into eq. (2.12), the power counting shows that the difference  $\check{S}_{CT}^{(pt)}[a_i, B_i] - \check{S}_{CT}^{(pt)}[a_i, B_i]|_{\mathcal{M}=0}$  may be associated only with a possible renormalization  $\delta^{(pt)} M_a^2$  of the mass of the field  $a_i$ . In turn,  $\delta^{(pt)} M_a^2 = 0$  by virtue of eq. (6.1).

To prove eq. (6.1), consider the generating functional  $\check{\mathcal{W}}_1[J_i|\mathcal{V}_i]$  which results after the averaging of the source  $e^{\int d^D x \text{Tr}(J_i a_i)}$  in the auxiliary high-energy theory (of the single dynamical field  $a_i$ ) defined by the action  $\check{S}[a_i - \mathcal{V}_i, \mathcal{V}_i]$  depending on the external field  $\mathcal{V}_i$  so that the normalization is chosen to be  $\check{\mathcal{W}}_1[0|0] = 0$ . Similarly to [5], one justifies that the  $\mathcal{V}_i = C_i$  option of the associated LEA  $\check{\Gamma}[C_i|\mathcal{V}_i]$  (with  $C_i$  being conjugated to  $J_i$ ) is the  $C_0 = 0$  reduction of a gauge-invariant functional:  $\check{\Gamma}[C_i|C_i] = \hat{\Gamma}[C_i]$  with  $\hat{\Gamma}[F_\mu] = \hat{\Gamma}[F_\mu^{(\omega)}]$ . Owing to the latter property of  $\check{\Gamma}[C_i|C_i]$ , it is the leading term  $\int d^D x \text{tr}(F_{\mu\nu}^2(C_i))/4\tilde{g}_r^2(\mathcal{M}_a/\Lambda_{YM})$  of the operator expansion of  $\check{\Gamma}[C_i|C_i]$  that defines the coupling constant  $\tilde{g}_r^2(\mathcal{M}_a/\Lambda_{YM})$ . In turn, to evaluate  $\mathcal{M}_a$ , one notes that the leading term of the operator expansion of  $\check{\Gamma}^r[C_i|0]$  assumes the form  $\mathcal{M}_a^2 \int d^D x \text{tr}(C_i^2)/2\tilde{g}_r^2(\mathcal{M}_a/\Lambda_{YM})$ . Altogether, the condition (6.1) follows (in view of  $C_i = C_i^r$ ) from the  $\mathcal{V}_i = C_i$  option of the relation

$$\check{\Gamma}^r[C_i|\mathcal{V}_i] = \check{\Gamma}^r[C_i|0] + \mathcal{M}^2 \int d^D x \text{Tr}(\mathcal{V}_i^2 - 2C_i \mathcal{V}_i)/2g_r^2(\Lambda/\Lambda_{YM}), \quad (6.3)$$

following from the fact that, in the action  $\check{S}[a_i - \mathcal{V}_i, \mathcal{V}_i]$ , the coupling between  $\mathcal{V}_i$  and  $a_i$  is linear in  $a_i = a_i^r$ . Indeed, in the theory defined by  $\check{S}[a_i - \mathcal{V}_i, \mathcal{V}_i]$ , consider the renormalized perturbative expansion in  $g_r(\tilde{M}/\Lambda_{YM})$  with some  $\tilde{M} > \Lambda_{YM}$  so that, at the tree-level, the  $a_i$ -mass is equal to  $\tilde{\mathcal{M}}_a = \mathcal{M}\tilde{g}_r(\tilde{M}/\Lambda_{YM})/g_r(\Lambda/\Lambda_{YM})$ , where  $\tilde{\Lambda}(\Lambda)$  is defined after eq. (6.2). Given the definition (6.1) of  $\mathcal{M}_a$ , it is the choice  $\tilde{M} = \tilde{\Lambda}^{-1}(\mathcal{M}_a)$  (resulting in  $\tilde{\mathcal{M}}_a = \mathcal{M}_a$ ) that, in view of eq. (6.3), allows to fix the *IR limit* both of the  $a_i$ -mass and of  $\tilde{g}_r^2$  already in the tree-approximation to the action (2.12). In this case, the  $n \geq 1$  loop contributions to the coefficient of the leading term of the operator expansion of  $\check{\Gamma}[C_i|C_i]$  are exactly cancelled by the counterterms (5.2).

<sup>7</sup>This identity follows from eq. (4.2) and the relation  $\Gamma^r[C_i|B_\mu] = \check{\Gamma}^r[C_i|B_\mu] + \int d^D x \text{Tr}(C_j \mathcal{T}_j(B))$  introduced prior to eq. (4.2).

## 7 Conclusions

Building on the insertion of the unity (3.1), we propose the multi-scale decomposition (1.2) which respects the background gauge invariance (1.5) and resolves, via eq. (4.2), the constraint (1.3) up to any given order of the loop-wise expansion. Choosing the axial gauge (1.6) and employing the ansatz (2.7)/(4.3), it introduces a novel renormalizable representation of the gauge theory (1.1). In turn, it allows to synthesize qualitatively different methods to evaluate the contribution of the high- and low-energy fields  $a_\mu$  and  $B_\mu$  interpolated at a scale  $\Lambda_{int}$ . The first average  $\langle \dots \rangle_a^B$  is performed employing the  $1/N$  weak-coupling expansion associated with a normalization point  $\Lambda$ . Due to the presence of the  $a_i$ -mass term (2.12) at the tree-level, it does not exhibit spurious  $IR$  singularities (present when this expansion is applied directly to the original formulation (1.1)) provided the  $IR$  limit  $\mathcal{M}_a(\mathcal{M}, \Lambda, \Lambda_{YM}) = \Lambda_{int}$  of the renormalized mass, defined by eq. (6.1), is sufficiently larger than  $\Lambda_{YM}$ . Integrating over  $a_\mu$ , an arbitrary correlator  $\mathcal{Q}[A_\mu] = \mathcal{Q}[A_\mu^{(\omega)}]$  is expressed, by virtue of eq. (2.5), in terms of gauge-invariant generically non-local correlators (i.e., Wilson loops with various operator's insertions) averaged with the  $B_\mu$ -dependent effective action (1.4). Respecting the gauge symmetry, the latter action is verified to be of the Wilsonian type. We also note that application of analytical approximations to the computation of  $S_{eff}[B]$  is considerably facilitated by the judicious adjustment (6.2) between the parameters  $\mathcal{M}_a$  and  $\Lambda$  when the proposed Ansatz depends (in addition to  $\Lambda_{YM}$ ) on the *single* parameter  $\mathcal{M}_a = \mathcal{M}$ .

As the low-energy theory is supposed to be strongly coupled, one possible way to evaluate the low-energy correlators is to develop further the stringy representation of the  $1/N$  strong-coupling expansion introduced in [2] (see also [3]) for the continuous  $D = 4$  Yang-Mills theory. The interpolation between the  $1/N$  strong- and  $1/N$  weak-coupling series suggests that the gauge theory can be represented in a synthetic way<sup>8</sup> combining "massive" gluons (with *two* propagating components) and fluctuating confining strings. Indeed, let identify  $e^{\mathcal{R}[A_\mu]}$  in eq. (1.2) with a macroscopic Wilson loop  $W_C$ . Then, appropriate segments of the base-contour  $C$  are collected, together with the trajectories of the  $a_\mu$ -gluons, into *closed* auxiliary contours  $\mathcal{C}_k$  which constitute boundaries of strings associated with the representation of the correlators like  $\langle \prod_k W_{\mathcal{C}_k} \rangle_B$ . A work in this direction is in progress.

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<sup>8</sup>It can be compared to the output of the semi-phenomenological approach [6] (corresponding to an implementation of the trick (3.1) with some unspecified  $a_\mu$ -independent action  $\tilde{S}_m[a_\mu, B_\mu] = \tilde{S}_m[B_\mu]$ ) where it is argued that massless, at least at the tree-level, gluons are coupled to a "frozen" string.